* Example: Binary Source

Source alphabet has symbols.

$S = \{s_0, s_1\}$

Source alphabet

$Pr(s_0) = p$

$Pr(s_1) = 1 - p$.

$$H(S) = Pr(S_0)\log_2\left(\frac{1}{Pr(S_0)}\right) + Pr(S_1)\log_2\left(\frac{1}{Pr(S_1)}\right)$$

$$H(S) = \underbrace{p\log_2\left(\frac{1}{p}\right) + (1-p)\log_2\left(\frac{1}{1-p}\right)}_{H(p)}$$

Entropy of Binary Source.

$$H(1-p) = (1-p)\log_2\left(\frac{1}{1-p}\right)$$

$$+(1-(1-p))\log_2\frac{1}{1-(1-p)}$$

$$= (1-p)\log_2\left(\frac{1}{1-p}\right) + p\log_2\left(\frac{1}{p}\right).$$

$$H(1-p) = H(p).$$

$$\lim_{p \to 0} H(p)$$

$$= \lim_{p \to 0} p \log_2 \left(\frac{1}{p}\right) + (1-p) \log_2 \left(\frac{1}{1-p}\right)$$

$(1-p) \to 1$, $\log_2 \left(\frac{1}{1-p}\right) \to 1$

$$= \lim_{p \to 0} \frac{p \log_2 \left(\frac{1}{p}\right)}{} + 1 \cdot \log_2 1$$

$\log_2 \left(\frac{1}{p}\right) \to \infty$, $\log_2 1 \to 0$

$$= 0$$

$$\boxed{\lim_{p \to 0} H(p) = 0}$$

$$H(p) = H(1-p)$$

$$\Rightarrow \lim_{p \to 1} H(p) = \lim_{p \to 1} H(1-p)$$

$$= \lim_{p \to 0} H(p)$$

$$\boxed{\lim_{p \to 1} H(p) = 0}$$

$p = 0 \Rightarrow Pr(S_0) = 0$

$Pr(S_1) = 1$

Source always generates S_1.

$p = 1 \Rightarrow Pr(s_0) = 1$

$Pr(s_1) = 0$

$\Rightarrow$ Source always generates s_0.

In both these cases, there is No uncertainty $\Rightarrow$ No information

$\Rightarrow$ Average Information per symbol $= 0$.

$$H(p) = p \log_2\left(\frac{1}{p}\right) + (1-p)\log_2\left(\frac{1}{1-p}\right)$$

To find maximum, let us differentiate w.r.t p.

$$H'(p) = \frac{d}{dp} \cdot H(p)$$

$$= \frac{d}{dp}\left(-p \log_2 p - (1-p)\log_2(1-p)\right)$$

$\log_e p \cdot \log_2 e$

$$= -\log_2 p - p \cdot \frac{1}{p} \cdot \log_2 e$$

$$+ \log_2(1-p) - (1-p) \cdot \frac{(-1)}{(1-p)} \log_2 e$$

$$= -\log_2 p - \cancel{\log_2 e} + \log_2(1-p) + \cancel{\log_2 e}$$

$$= -\log_2 p + \log_2 (1-p)$$

$$\equiv 0$$

To find the maximum

$$\Rightarrow \log_2 \frac{1-p}{p} = 0.$$

$$\Rightarrow \frac{1-p}{p} = 1$$

$$\Rightarrow 1-p = p$$

$$\Rightarrow \boxed{p = \frac{1}{2}}$$

$$Pr(S_0) = Pr(S_1) = \frac{1}{2}$$

$$H''(p) = \frac{d}{dp} H'(p)$$

$$= \frac{d}{dp}\left(-\log_2 p + \log_2(1-p)\right)$$

$$= \log_2 e \cdot \left(-\frac{1}{p} - \frac{1}{1-p}\right)$$

$$= -\log_2 e \left(\frac{1}{p} + \frac{1}{1-p}\right)$$

$p \geq 0 \cdot 1-p \geq 0$

≤ 0

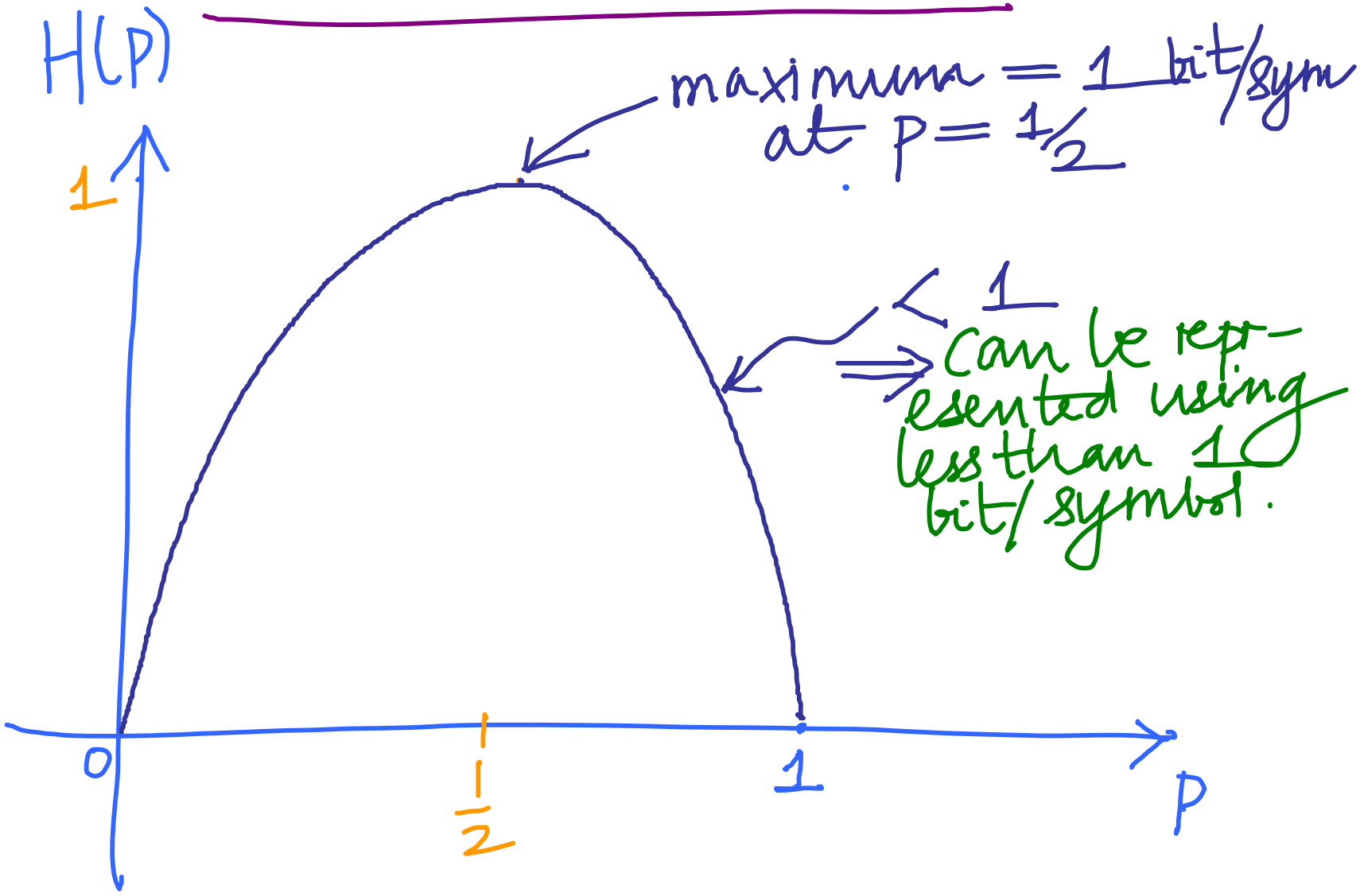
$\Longrightarrow$ maximum at $p = \frac{1}{2}$

$$H(p=\tfrac{1}{2}) = \tfrac{1}{2}\log_2 2 + \tfrac{1}{2}\cdot\log_2 2$$

$$\boxed{H(\tfrac{1}{2}) = \tfrac{1}{2} + \tfrac{1}{2} = 1 \text{ bit/symbol}.}$$

$$\boxed{H(0) = H(1) = 0.}$$

Plot H(p):


H(p)
1
maximum = 1 bit/sym
at p = 1/2
1
⟹ can be represented using less than 1 bit/symbol.
0
1/2
1
p